

State–Space Formulation of Linear Systems

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Developing noise and vibration propagation models is an important task in designing solutions to the noise and vibration control problems. Two modeling and analysis approaches are commonly used in linear systems, frequency response function (FRF) and state space formulation. Frequency response function which can be viewed as a subset of transfer function has been the modeling technique of choice by engineers in the area of noise and vibration. As convenient as FRF modeling is, it is not the most suitable method for modeling multi-variable systems and designing controllers for them.

State space modeling, which is just another way of presenting differential equations describing a dynamic system. It uses a set of 1st order differential equations. This modeling method may appear novel to practitioners who are accustomed to thinking in terms of frequency response function (or transfer function) but it is not a new way of looking at dynamic systems. Physicists and control engineers have been using this modeling technique for years. State space modeling enables noise and vibration engineers to have access to and put to use a wealth of knowledge and analysis techniques from the linear systems discipline, including designing estimators and controllers for multi–input–multi–output systems.

1 State Space Models

Perhaps the best way to describe state space modeling is by an example. The following is a simple mass, spring, dashpot system with a force input $f(t)$ and position $x(t)$.

The differential equation describing this system is

$$m\ddot{x}(t) + c\dot{x} + kx = f(t) \quad (1)$$

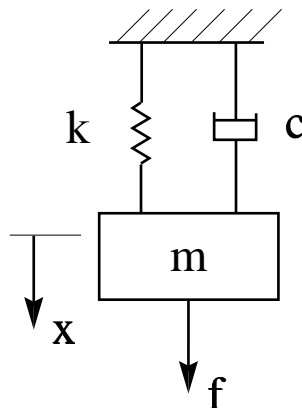


Figure 1: A simple mass–spring–dashpot system

And the transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \quad (2)$$

Since we have a second order system, we will need two states to describe it. Although the states of a system are not unique, displacement and velocity of the mass m are the most suitable choice of states in this system. We define the state variables $x_1(t)$ and $x_2(t)$ as:

$$x_1(t) = x(t) \quad \text{position} \quad (3)$$

$$x_2(t) = \dot{x}(t) \quad \text{velocity} \quad (4)$$

Equation 1 will be written in terms of the state variables, their first derivatives, and the input $f(t)$; see Equation 5. No second derivatives will be present.

$$m\dot{x}_2(t) = -cx_2(t) - kx_1(t) + f(t) \quad (5)$$

In addition to the state equation, i.e., Equation 5, an algebraic equation will be used to describe the output of interest, e.g. the displacement of the mass. The output is a linear combination of states and inputs.

We now have the equations we need to put the system into state space form. On the left hand side of the state equations, we place only the first derivatives of the state variables; see Equations 6 and 7. And as shown in Equation 9, the output variable(s) is(are) placed in the left hand side of the output equation.

$$\dot{x}_1(t) = x_2(t) \quad (6)$$

$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}f(t) \quad (7)$$

$$(8)$$

$$y(t) = x_1(t) \quad (9)$$

When written in standard matrix format the equations are as follows:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t) \quad (10)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (11)$$

The generic form for state equations is given by equations 12 and 13.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (12)$$

$$y(t) = Cx(t) + Du(t) \quad (13)$$

where

$x(t)$ = $(n \times 1)$ state vector where n is the number of states or system order

$u(t)$ = $(r \times 1)$ input vector where r is the number of input functions

$y(t)$ = $(p \times 1)$ output vector where p is the number of outputs

A = $(n \times n)$ square matrix called the system matrix

B = $(n \times r)$ matrix called the input matrix

C = $(p \times n)$ matrix called the output matrix

D = $(p \times r)$ matrix which represents any direct connection between the input and output

The transfer function equivalent of Equations 12 and 13 can be derived by taking the Laplace transformation of Equation 12 and substituting it in Equation 13, resulting in

$$Y(s) = \{C[sI - A]^{-1}B + D\}U(s) \quad (14)$$

Note that $\{C[sI - A]^{-1}B + D\}$ is the transfer function matrix mapping the input vector $U(s)$ to the output vector $Y(s)$. Moreover, the poles of this transfer function are the same as the eigenvalues of the A matrix of the state space representation of the same system.

Example 1.1: Modeling of a simple system

For $m = 2$ Kg, $c = 1.5$ Nsec/m, and $k = 100$ N/m, find the poles of the second order system of Figure 1.

Equating the denominator of the transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{2s^2 + 1.5s + 100}$$

to zero results in the characteristic equation, the solution of which are the poles of the system.

$$s = -0.375 \pm j7.061$$

The eigenvalues of the A matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{100}{2} & -\frac{1.5}{2} \end{bmatrix}$$

are also the poles of the system

$$eig(A) = -0.375 \pm j7.061$$

poleexa

The solution of the state Equations 12, comprising of the homogeneous and forced parts, is

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Bu(\tau)d\tau \quad (15)$$

where $\Phi(t)$ is known as the state transition matrix and is defined as

$$\Phi(t) = e^{At} = \mathcal{L}^{-1}([sI - A]^{-1}) \quad (16)$$

and \mathcal{L}^{-1} indicates the inverse Laplace transformation.

1.1 Formal Definitions

State: The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \leq t_0$, completely determines the behavior of the system for any time $t \geq t_0$. The state of a dynamic system at time t is uniquely determined by the state at time t_0 and the input for $t \leq t_0$, and it is independent of the state and input before t_0 . Frequently, in linear time-invariant systems, we choose the initial time t_0 to be zero.

The concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

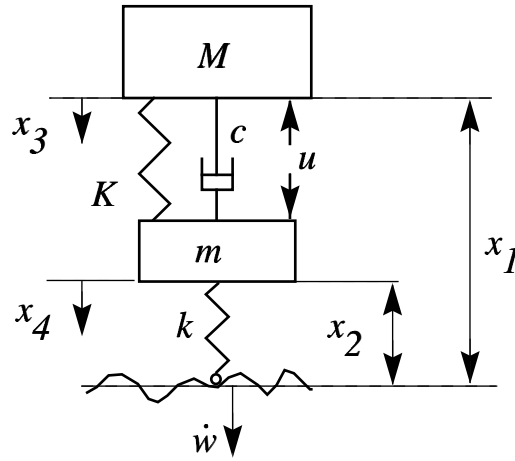


Figure 2: Automobile active suspension system

State variables: The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables x_1, x_2, \dots, x_n are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \leq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Note that the state variables of a system are not unique. Moreover, the state variables need not be physically measurable¹ quantities. Variables that do not represent physical quantities and those that are not measurable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state space methods.

State vector: If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector x . Such a vector is called a state vector. A state vector is thus a vector that determines uniquely the system state $x(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State space: The n -dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, \dots , x_n axis is called a state space. Any state of a can be represented by a point in the state space.

Example 1.2: Active Suspension System

Quarter car model of an active suspension system, depicted in Figure 2, is used to illustrate the state space formulation.

The states x_1 through x_4 , shown in Figure 2, are the displacements of the two masses relative to the road unevenness w and their absolute velocities. These are the most common set of states used in active suspension literature. $x = [x_1, \dots, x_4]^T$ is the state vector of the system.

Working space $(x_1 - x_2)$ is the measured output. The rate of change of road unevenness \dot{w} , i.e., the disturbance, and the control force u are the inputs to the system.

$$\begin{aligned} \dot{x} &= Ax + \left[\begin{array}{cc} B_1 & B_2 \end{array} \right] \left\{ \begin{array}{c} \dot{w} \\ u \end{array} \right\} \\ &= Ax + B_2 u + B_1 \dot{w} \end{aligned}$$

¹Modal displacement and velocity are good examples of unmeasurable state variables.

$$y = Cx + Du$$

The quarter car suspension has the following state space realization:

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/M & K/M & -c/M & c/M \\ K/m & -(K+k)/m & c/m & -c/m \end{bmatrix} \\
 B &= [B_1 \ B_2] \\
 B_2 &= [0 \ 0 \ -\frac{1}{M} \ \frac{1}{m}]' \\
 B_1 &= [-1 \ -1 \ 0 \ 0]' \\
 C &= [1 \ -1 \ 0 \ 0] \\
 D &= [0]
 \end{aligned}$$

asexa

1.2 Why Use State Space

There are many advantages to modeling systems in state space. The most important advantage is the matrix formulation. Computers can easily manipulate matrices. Having the A , B , C , and D matrices, one can calculate stability, controllability, observability and many other useful attributes of a system. The second most important aspect of state space modeling is that it allows us to model the internal dynamics of the system, as well as the overall input/output relationship as in transfer functions. And, as stated earlier, state space modeling makes the vast, existing, linear system knowledge such as estimation and optimal control theory available to the user.