Modal Analysis Theory

R. Kashani, Ph.D.
www.deicon.com

Modal Analysis is a computationally elegant technique for modeling structural dynamics. It is based on the use of eigenvalue and Eigenvector information of a system. The elegance and appeal of this technique is mainly due to its decoupling capability. Moreover, it is the basis for understanding modal test methods (known also as experimental modal analysis).

Considering the forced response of an underdamped system shown by Equation 1

$$M\ddot{x}(t) + Kx(t) = f(t) \tag{1}$$

where M and K are $n \times n$ positive definite matrices of mass and stiffness, $x(t)$ is the vector of displacement, and $f(t)$ is the vector of applied force. Equation 1 can be solved usinjg eigenvalue expansion. This is done by first solving the eigenvalue problem for the corresponding homogenous system

$$A\psi = \lambda\psi \tag{2}$$

where $A = M^{-1}K$, λ is the diagonal matirx of eigenvalues, and ψ is the matrix of eigenvectors. The spatial coordiante x can be changed to a new coordinate η using

$$\eta = \psi^{-1}x \tag{3}$$

where ψ is an invertible matrix. Premultiplying Equation 3 by ψ results in

$$x = \psi\eta \tag{4}$$

Substituting for x in Equation 1 from Equation 4

$$M\psi\ddot{\eta} + K\psi\eta = f(t) \tag{5}$$

Premultiplying Equation 5 by ψ^{-1}

$$\psi^{-1}M\psi\ddot{\eta} + \psi^{-1}K\psi\eta = \psi^{-1}f(t) \tag{6}$$

Using similarity transformation (described in the following section)

$$\psi^{-1} = \psi^T$$

Moreover, $\psi^T M\psi$ and $\psi^T K\psi$ are diagonal matrices.

Mass normalizing the eigenvectors, i.e.,

$$\psi^T M\psi = I$$

will result in

$$\psi^T K \psi = \mathrm{d}iag(\omega_n^2)$$

Thus the use of mass normalized eigenvectors as the columns of the transformation matrix ϕ, results in a set of n decoupled 2nd order differential equations of

$$\ddot{\eta}_i + \omega_i^2 \eta_i = f_i(t) \tag{7}$$

where f_i denotes the i-th element of the vector $\psi^T f$.

Equation 4 indicates that the physical displacement of the structure, x, is the summation of its modal contributions, i.e., modal displacements η_i's scaled by their corresponding eigenvector (mode shape). In other words

$$x(t) = \sum_{i=1}^{n} \psi_i \eta_i \tag{8}$$

1 Similarity Transformation

Two matrices A and B are similar if they have the same eigenvalues. Their transformation is via a nonsingular matrix P, according to

$$A = P^{-1} B P$$

When the columns of the similarity transformation P consist of the n eigenvectors of B, the similar matrix of B becomes diagonal, denoted by Λ, i.e.,

$$\Lambda = P^{-1} B P$$

or

$$B = P \Lambda P^{-1}$$

This can be rewritten as

$$BP = P\Lambda \tag{9}$$

The matrix equation 9 is known as eigenvalue problem. If λ_{ii} denotes the i-th diagonal element of the diagonal matrix Λ, Equation 9 can be rewritten as n seperate equations

$$AP_i = \lambda_{ii} P_i \qquad i = 1, 2, \ldots, n \tag{10}$$

Equation 10 states that P_i is the i-th eigenvector of the matrix P and λ_{ii} is the associated eigenvalue.

2 Poles and Zeros

Consider the equation of motion in modal domain, i.e.,

$$\ddot{\eta}(t) + \omega_n^2 \eta(t) = \psi^T f(t) \tag{11}$$

$$x = \psi \eta \tag{12}$$

Taking the Laplace transform of Equations 11 and 12 results in

$$
\begin{aligned}
[s^2+\omega_n^2]\eta(s) &= \psi^T f(s) && (13)\\
x(s) &= \psi\eta(s) = \psi[s^2+\omega_n^2]^{-1}\psi^T f(s) && (14)
\end{aligned}
$$

leading to the transfer function matrix mapping the input f to the output x

$$\psi[s^2+\omega_n^2]^{-1}\psi^T f(s)$$

and the frequency response function (FRF) of

$$
\begin{aligned}
\alpha(\omega) &= \psi[-\omega^2+\omega_n^2]^{-1}\psi^T \\
&= \psi[\omega_n^2-\omega^2]^{-1}\psi^T && (15)
\end{aligned}
$$

Note that the term $[\omega_n^2-\omega^2]^{-1}$ in Equation 15 is a diagonal matrix. The FRF of Equation 15 can also be written in summation notation by considering the *ik*-th element of $\alpha(w)$ and partitioning the matrix ψ into columns denoted by ψ_r. The vectors ψ_r are the mass normalized modal vectors, i.e., eigenvectors of the matrix K normalized with respect to the mass matrix M. This yields

$$\alpha(\omega) = \sum_{r=1}^{n}[\omega_r^2-\omega^2]^{-1}\psi_r\psi_r^T \tag{16}$$

The *ik*-th element of the $\alpha(\omega)$ matrix becomes

$$\alpha_{ik}(\omega) = \sum_{r=1}^{n}[\omega_r^2-\omega^2]^{-1}[\psi_r\psi_r^T]_{ik} \tag{17}$$

where the matrix elements $[\psi_r\psi_r^T]_{ik}$ is identified as the modal constant or *residue* for the *r*-th mode and the matrix $[\psi_r\psi_r^T]$ is called the residue matrix.

Partial Fraction Expansion

The 2nd. order system of Equation 18 can be expressed as the sum of partial fractions shown in Equation 19.

$$
\begin{aligned}
H(s) &= \frac{\frac{1}{M}}{(s-\lambda_1)(s-\lambda_1^*)} && (18)\\
&= \frac{c_1}{(s-\lambda_1)} + \frac{c_2}{(s-\lambda_1^*)} && (19)
\end{aligned}
$$

where $\lambda_1 = \sigma_1 + j\omega_1$ and $\lambda_1^* = \sigma_1 - j\omega_1$ are the complex conjugate poles and c_1 and c_2 defined below are the complex conjugate residues of the rationoal polynomial transfer function describing the 2nd order system.

$$
\begin{aligned}
c_1 &= \frac{\frac{1}{M}}{j2\omega_1} && (20)\\
c_2 &= -\frac{\frac{1}{M}}{j2\omega_1} = c_1^* && (21)
\end{aligned}
$$

The mathematical representation of a transfer function in terms of a partial fraction expansion in nothing more than a sum of single degree of freedom systems. Therefore the mass of a single dof system, which is by definition the *modal mass*, canbe related to the residue for a single dof system.

Note that the in the jargons of multi-dof system, the transfer function of a single dof system, i.e., Equation 18, is a driving point (collocated) transfer function. Considering that a multi-dof transfer function, or frequency response function, is represented as $H(s)_{11}$ or $H(w)_{11}$, the residue c_1 which is the driving point residue will be represented as c_{qq1}

The modal mass for a single dof system is

$$\begin{aligned} M &= \frac{1}{j2\omega_1 c_1} \\ &= \frac{1}{j2\omega_1 c_{qq1}} \\ &= -\frac{j}{2\omega_1 c_{qq1}} \end{aligned} \tag{22}$$

Recalling that the residue matrix for a particular pole λ_r is related to the modal vectors. For an r–th mode of an N dof system

$$\begin{aligned} [c]_r &= Q_r\{u\}_r\{u\}_r^T \\ &= Q_r \begin{bmatrix} u_1u_1 & u_1u_2 & \dots & u_1u_m \\ u_2u_1 & u_2u_2 & & \\ \vdots & & \ddots & \\ u_mu_1 & & & u_mu_m \end{bmatrix}_r \end{aligned} \tag{23}$$

where Q_r is an arbitrary scaling constant. Now the r-th modal mass of a multi-dof is defined as

$$M_r = \frac{1}{j2Q_r\omega_r} \tag{24}$$

Example 2.1: A 2-dof system
Using the differential Equation 25,

$$\begin{aligned} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} &+ \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ &= \begin{Bmatrix} F \\ 0 \end{Bmatrix} \end{aligned} \tag{25}$$

the model of the 2dof undamped system shown in Figure 1 in the absence of damping is given in Equation 26

$$M\ddot{x}(t) + Kx(t) = u(t) \tag{26}$$

where $x = [x_1 \quad x_2]'$ and $u = [f_1 \quad f_2]'$ are the vectors of displacements (outputs) of the two masses and forces (inputs) exciting the two masses; see Figure ???. The transfer function matrix mapping the input vector to output vector is evaluated by taking the Laplace transfrom of Equation 26, resulting in

$$[Ms^2 + K]x(s) = u(s) \tag{27}$$

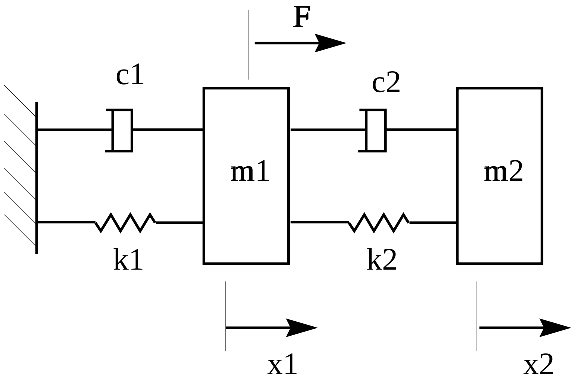

Figure 1: A 2–DOF discrete system

Using the differential Equation 25

$$
\begin{aligned}
x(s) &= [Ms^2+K]^{-1}u(s) \\
&= \left\{ \begin{bmatrix} m_1s^2 & 0 \\ 0 & m_2s^2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \right\}^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} & (28) \\
&= \begin{bmatrix} m_1s^2+k_1+k_2 & -k_2 \\ -k_2 & M_2s^2+K_2 \end{bmatrix}^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} & (29) \\
&= \underbrace{\frac{\begin{bmatrix} m_2s^2+k_2 & k_2 \\ k_2 & m_1s^2+k_1+k_2 \end{bmatrix}}{\det}}_{G(s)} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} & (30)
\end{aligned}
$$

where

$$\det = m_1m_2s^4 + (m_1k_2 + m_2k_1 + m_2k_2)s^2 + k_1k_2$$

is the determinant of the matrix $[Ms^2+K]$. Letting $m_1 = 8$, $m_2 = 15$, $k_1 = 5$, and $k_2 = 10$, the transfer function $G(s)$ in Equation 30 becomes

$$
\begin{aligned}
G(s) &= \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \\
& \frac{\begin{bmatrix} 15s^2+10 & 10 \\ 10 & 8s^2+15 \end{bmatrix}}{120s^4+305s^2+50} & (31)
\end{aligned}
$$

Looking at the first element of the transfer function matrix 31,

$$
\begin{aligned}
G_{11}(s) &= \frac{15s^2+10}{120s^4+305s^2+50} \\
& \frac{-j0.0315}{s-j1.538} + \frac{j0.0315}{s+j1.538} + \frac{-j0.0334}{s-j0.4197} + \frac{j0.0334}{s+j0.4197} & (32)
\end{aligned}
$$

Note that the pairs $c_{111} = -j0.0315$, $c^*_{111} = j0.0315$ and $c_{112} = -j0.0334$, $c^*_{112} = j0.0334$ are complex conjugate residues corresponding to the two complex conjugate pairs of poles.

In a similar fashion the rest of the residues for the remaining transfer functions can be determined. The system transfer function $G(s)$ can now be expressed in terms of partial fractions, shown in Equation 33.

$$
\begin{aligned}
G(s) = \quad & \frac{\begin{bmatrix} -0.0315i & 0.0124i \\ 0.0124i & -0.0049i \end{bmatrix}}{s - 1.5380i} + \frac{\begin{bmatrix} 0.0315i & -0.0124i \\ -0.0124i & 0.0049i \end{bmatrix}}{s + 1.5380i} + \\
& \frac{\begin{bmatrix} -0.0334i & -0.0453i \\ -0.0453i & -0.0616i \end{bmatrix}}{s - 0.4197i} + \frac{\begin{bmatrix} +0.0334i & +0.0453i \\ +0.0453i & +0.0616i \end{bmatrix}}{s + 0.4197i}
\end{aligned}
\tag{33}
$$

Note that residues corresponding to a pair of complex conjugate poles, appear as a complex conjugate pair themselves. Recall that the modal vector associated with each pole is proportional to the residue matrix for that pole. This makes the modal vectors corresponding to a pair of complex conjugate poles complex conjugate of each other, as well.

Relating the modal vector to the residue matrix of the first pole

$$
\begin{aligned}
Q_1\{u\}_1\{u\}_1' &= Q_1 \begin{bmatrix} u_1u_1 & u_1u_2 \\ u_2u_1 & u_2u_2 \end{bmatrix}_1 \\
&= \begin{bmatrix} -0.0315i & 0.0124i \\ 0.0124i & -0.0049i \end{bmatrix} \\
&= 100 \begin{bmatrix} -3.15i & 1.24i \\ 1.24i & -0.49i \end{bmatrix}
\end{aligned}
\tag{34}
$$

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3 Modal Vector Scaling

One of the widely used scaling techniques of modal vectors is "unity modal mass" in which the scaling factor Q_r is decided upon based on having unity modal mass M_r, i.e.,

$$
\begin{aligned}
Q_r &= \frac{1}{j2M_r\omega_r} \\
&= \frac{1}{j2\omega_r} \\
&= -\frac{j}{2\omega_r}
\end{aligned}
\tag{35}
$$

Using the unity modal mass scaling factor in conjunction with the diagonal elements (corrensponding to driving force measurements) of Equation 23, the scaled modal coefficients can be computed as

$$u_{qr}u_{qr} = \frac{c_{qqr}}{Q_r} \tag{36}$$

Having the driving point scaled modal coefficients (corresponding to the diagonal elements of coefficient matrix in Equation 23), the other coefficients (off–diagonal elements of the matrix can be calculated. In general, the scaled modal vector is

$$\{u\}_r = \frac{1}{Q_r u_{qr}}\{c\}_r \tag{37}$$